# Laplacians are equivalent to Simplices: an invitation

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**Abstract**

We work out a small concrete example of the equivalence among connected weighted graphs, Laplacians, and Simplices, as described by Devriendt in a review of Fiedler's work.

In a 2022 review article, Devriendt [1] surveys some of Fiedler's work at the intersection of graph theory, linear algebra, and geometry. It is, in some ways, an invitation to Fiedler's work. Here we attempt an invitation to Devriendt's review article, by working out for a small concrete example the details of the fundamental equivalence discussed therein. We will define the types of objects we will be relating as we go.

**Definition 1.** A *weighted graph* on $n$ vertices consists of a finite, connected simple graph $G = (V, E)$ with $|V| = n$ together with a *weight function* $\omega \colon E \to \mathbb{R}^+$.

**Example 2.** In Figure 1 we display a graph $G$ on 4 vertices. For simplicity, we have chosen to give the edges of $G$ the constant weight 1.

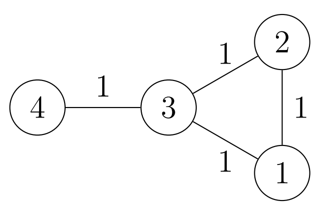


Figure 1: A weighted graph $G$ with constant weight 1.

**Definition 3.** An $n \times n$ *Laplacian* is a matrix $Q \in \mathbb{R}^{n\times n}$ such that:

1. $Q$ is symmetric;
2. the off-diagonal entries of $Q$ are non-positive;
3. the rows and columns of $Q$ sum to zero; and
4. $Q$ is irreducible, that is, it is not permutation similar to a block triangular matrix.

Weighted graphs on $n$ vertices and $n \times n$ Laplacians are in bijective correspondence. Given a weighted graph $G$ on $n$ vertices, its corresponding Laplacian $Q = Q(G)$ is the $n \times n$ matrix with entries defined by the expression

$$Q_{ij} = \begin{cases} \sum_{\ell \in N(i)} \omega_{i\ell} & i = j \\ -\omega_{ij} & j \in N(i) \\ 0 & \text{otherwise.} \end{cases}$$

The weighted graph $G$ can be recovered readily from $Q$. The number of rows of $Q$ gives the number of vertices, two vertices $i$ and $j$ are connected by an edge according to whether $Q_{ij}$ is nonzero, and the negation of the weight is directly displayed on off-diagonal entries.

**Example 4.** The Laplacian corresponding to the weighted graph $G$ from Example 2 is

$$Q = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The Laplacian $Q$ has the spectral decomposition

$$Q = \sum_{\ell=1}^{4-1} \mu_\ell z_\ell z_\ell^T,$$

where

$$\begin{aligned} \mu_1 &= 4 & z_1 &= \begin{bmatrix} 1 & 1 & -3 & 1 \end{bmatrix}^T \\ \mu_2 &= 3 & z_2 &= \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix}^T \\ \mu_3 &= 1 & z_3 &= \begin{bmatrix} -1 & -1 & 0 & 2 \end{bmatrix}^T \\ \mu_4 &= 0 & z_4 &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T. \end{aligned}$$

By property 3 of the Laplacian, the all-ones vector $\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}^T$ is always an eigenvector.

Recall that an $(n-1)$-*simplex* in $\mathbb{R}^{n-1}$ is the convex hull of $n$ affinely independent points in $\mathbb{R}^{n-1}$.

**Definition 5.** Let $\Sigma$ be the set of all $(n-1)$-simplices in $\mathbb{R}^{n-1}$. Define an equivalence relation $\sim$ on $\Sigma$ by

$$S \sim S' \iff S' = OS + xu^T,$$

for some orthogonal matrix $O$ and vector $x$. Then a (capital-S) *Simplex* is an equivalence class of this equivalence relation. Observe that by construction, representatives of a Simplex are congruent but might have different orientations and centroids.

A Simplex $\mathcal{S}$ has a canonical Gram matrix representative. Let $S \in \mathcal{S}$ be any representative simplex and define

$$M = M(\mathcal{S}) = (I - uu^T/n)^T S^T S(I - uu^T/n).$$

Observe that $M$ has its centroid at the origin and the value of $M$ does not depend on the choice of $S$.

While every Laplacian can be regarded as the canonical Gram matrix for a Simplex, a Gram matrix with centroid at the origin might not be non-positive on the off-diagonal.

**Definition 6.** A Simplex $\mathcal{S}$ is *hyperacute* if none of its dihedral angles are obtuse.

The canonical Gram matrix for a hyperacute Simplex has non-positive off-diagonals and thus corresponds to a Laplacian. Given a Laplacian $Q$, its corresponding Simplex is the equivalence class of the $(n-1)$-simplex with vertices $s_i$ with coordinates

$$(s_i)_k = (z_k)_i \sqrt{1/\mu_k}.$$

**Example 7.** Let $Q$ be the Laplacian from Example 4. Then its corresponding Simplex has a representative 3-simplex with vertices

$$\begin{aligned} s_1 &= (1/2, -1/\sqrt{3}, -1), \\ s_2 &= (1/2, 1/\sqrt{3}, -1), \\ s_3 &= (-3/2, 0, 0), \\ s_4 &= (1/2, 0, 2). \end{aligned}$$

This 3-simplex is displayed in Figure 2.

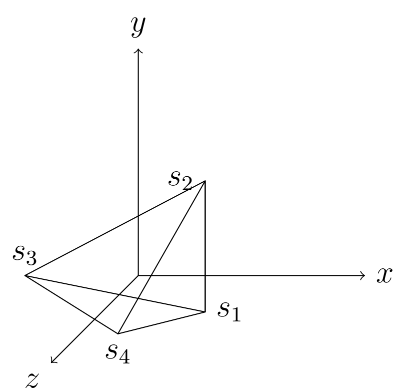


Figure 2: A 3-simplex corresponding to the graph $G$ and Laplacian $Q$.

Devriendt's review article discusses much more, but our goal here is only to give a taste via an extended example, and hopefully spark interest in the review article and in Fiedler's work.

# References

[1] Karel Devriendt. Effective resistance is more than distance: Laplacians, Simplices and the Schur complement. *Linear Algebra and its Applications*, 639:24–49, 2022.

[2] Reinhard Diestel. *Graph theory*. Graduate texts in mathematics ; 173. Springer, Berlin, fifth edition. edition, 2017 - 2017.

[3] Chris Godsil and Gordon Royle. *Algebraic graph theory*. Graduate texts in mathematics ; 207. Springer, New York, 2001.

[4] Roger A. Horn and Charles R. Johnson. *Matrix Analysis*. Cambridge University Press, 1985.