Absolute betweenness

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In this note, we verify a standard betweenness result about the absolute value. We define the absolute value function by $|x| = \sqrt{x^2}$. Hence in particular $|x|^2 = x^2$.

Theorem. *Let $x, y, z \in \mathbb{R}$. Suppose $|x-z| = |x-y| + |y-z|$. Then either $x \leq y \leq z$ or $z \leq y \leq x$.*

Proof. The result is straightforward if any pair of x, y, z coincide, so we may assume that x, y, z are distinct. Introduce the auxiliary variables $a = x - z$, $b = x - y$, and $c = y - z$, and observe that $|a| = |b| + |c|$ and $a = b + c$. We compute

$$\begin{aligned} 2ab &= a^2 + b^2 - (a-b)^2 \\ &= a^2 + b^2 - c^2 \\ &= |a|^2 + |b|^2 - |c|^2 \\ &= |a|^2 + |b|^2 - (|a| - |b|)^2 \\ &= 2|a||b|, \end{aligned}$$

that is,

$$(x-z)(x-y) = |x-z||x-y|. \tag{1}$$

By a similar computation, we obtain

$$(x-z)(y-z) = |x-z||y-z|. \tag{2}$$

From Equation 1 we deduce that $x-z$ and $x-y$ have the same sign, and from Equation 2 that $x-z$ and $y-z$ have the same sign. It follows that $x-y$ and $y-z$ have the same sign. If they are both positive, then $z < y < x$. If they are both negative, then $x < y < z$. $\square$