On the irrationality of the number $e = 2.718\ldots$

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One proves elementarily that the number e, the base of Napierian [natural] logarithms, does not have a rational value. We should, it seems to me, add that the same method also proves that e cannot be a root of a second-degree equation in rational coefficients, so that one cannot have $ae + \frac{b}{e} = c$, a being a positive integer and b, c positive or negative integers. Indeed, if one replaces in this equation e and $\frac{1}{e}$ or e^{-1} by their developments [power series expansions] deduced from that of e^x, then we multiply the two sides of the equation by $n! = 1 \cdot 2 \cdot 3 \cdots n$, we will easily find

$$\frac{a}{n+1}\left(1 + \frac{1}{n+2} + \cdots\right) \pm \frac{b}{n+1}\left(1 - \frac{1}{n+2} + \cdots\right) = \mu,$$

μ being an integer. We can always ensure that the factor

$$\pm\frac{b}{n+1}$$

is positive; it will suffice to assume n even if $b < 0$ and n odd if $b > 0$. By taking n to be very large, the equation that we have just written will lead to an absurdity, because its left-hand side, being essentially positive and very small, will be between 0 and 1, and cannot be equal to an integer μ. Therefore, etc.

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