The cartesian plane is a model of incidence geometry

MLE Slone

August 16, 2026

In this note we verify that the cartesian plane satisfies the axioms of incidence geometry. Here is the axiomatization we will use for incidence geometry. The terms *point*, *line*, and *lies on* are undefined. We say that points P, Q, and R are *collinear* if there exists a line ℓ such that P, Q, and R lie on ℓ, and say that they are *non-collinear* otherwise.

Axiom 1. *For each pair of distinct points P and Q, there is a unique line ℓ such that P and Q lie on ℓ.*

Axiom 2. *For each line ℓ, there exist distinct points P and Q such that P and Q lie on ℓ.*

Axiom 3. *There exist three distinct non-collinear points.*

Recall that $\mathbb{R}$ is the unique (up to isomorphism) complete ordered field.

Definition 4. *The* cartesian plane *is $\mathbb{R}^2$, the two-dimensional vector space over $\mathbb{R}$ with the standard basis vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$.*

Definition 5. *A* point *of the cartesian plane is a vector in $\mathbb{R}^2$.*

Definition 6. *A* line *in the cartesian plane is a subset of $\mathbb{R}^2$ of the form*

$$L(a, b, c) = \{(x, y) : ax + by + c = 0\},$$

where a, b, and c are in $\mathbb{R}$ and a and b are not both 0.

Definition 7. *A* point P lies on *a line L just when $P \in L$.*

Now that we have an interpretation, we will proceed to verify the axioms. We begin by establishing a canonical representation for lines.

Lemma 8. *Let $L(a, b, c)$ be a line. Then for any $\lambda \neq 0$,*

$$L(\lambda a, \lambda b, \lambda c) = L(a, b, c).$$

Proof. Let (x, y) lie on $L(a, b, c)$. Then

$$(\lambda a)x + (\lambda b)y + (\lambda c) = \lambda(ax + by + c) = \lambda \cdot 0 = 0,$$

so (x, y) lies on $L(\lambda a, \lambda b, \lambda c)$.

Conversely, suppose (x, y) lies on $L(\lambda a, \lambda b, \lambda c)$. Since $\lambda \neq 0$, its multiplicative inverse λ^{-1} exists. Thus

$$ax + by + c = (\lambda^{-1}\lambda a)x + (\lambda^{-1}\lambda b)y + (\lambda^{-1}\lambda c) = \lambda^{-1}((\lambda a)x + (\lambda b)y + (\lambda c)) = \lambda^{-1} \cdot 0 = 0,$$

so (x, y) lies on $L(a, b, c)$. $\square$

Proposition 9. *Each line can be written in the form $L(a,b,c)$ with $a=1$ or $b=1$.*

Proof. Let $L(a,b,c)$ be a line. If $a \neq 0$, by Lemma 8,

$$L(a,b,c) = L(a/a, b/a, c/a) = L(1, b/a, c/a).$$

If $a = 0$, then since a and b are not both 0, it follows that $b \neq 0$ and we may write

$$L(a,b,c) = L(a/b, b/b, c/b) = L(a/b, 1, c/b).$$

□

Proposition 10. *The cartesian plane satisfies Axiom 2.*

Proof. Let ℓ be a line. By Proposition 9, without loss of generality, we may write $\ell = L(1,b,c)$. It is straightforward to verify that $P = (-c, 0)$ and $Q = (-b-c, 1)$ are distinct points which lie on ℓ. □

Lemma 11. *Let P and Q be distinct points. Then there exists a line $L = L(a,b,c)$ such that P and Q lie on L.*

Proof. Suppose $P = (p_1, p_2)$ and $Q = (q_1, q_2)$. Then define

$$\begin{aligned} a &= q_2 - p_2 \\ b &= p_1 - q_1 \\ c &= p_2 q_1 - p_1 q_2. \end{aligned}$$

We compute

$$\begin{aligned} ap_1 + bp_2 + c &= (q_2 - p_2)p_1 + (p_1 - q_1)p_2 + (p_2 q_1 - p_1 q_2) \\ &= p_1 q_2 - p_1 p_2 + p_1 p_2 - p_2 q_1 + p_2 q_1 - p_1 q_2 \\ &= 0, \end{aligned}$$

so P lies on $L(a,b,c)$. Similarly,

$$\begin{aligned} aq_1 + bq_2 + c &= (q_2 - p_2)q_1 + (p_1 - q_1)q_2 + (p_2 q_1 - p_1 q_2) \\ &= q_1 q_2 - p_2 q_1 + p_1 q_2 - q_1 q_2 + p_2 q_1 - p_1 q_2 \\ &= 0, \end{aligned}$$

so Q also lies on $L(a,b,c)$. □

Lemma 12. *Let ℓ and m be lines. If $\ell \cap m$ contains two distinct points, then $\ell = m$.*

Proof. Using Proposition 9, we may write $\ell = L(1,b,c)$ and $m = L(d,e,f)$ with $d = 1$ or $e = 1$.

Let (x,y) be a point in the intersection $\ell \cap m$. Then (x,y) satisfies the following equations:

$$x + by + c = 0; \tag{1}$$

$$dx + ey + f = 0. \tag{2}$$

By subtracting Equation 2 from Equation 1, we obtain

$$(1-d)x + (b-e)y + (c-f) = 0. \tag{3}$$

Let us consider the $d = 1$ case. Assume that $b \neq e$. From this we conclude that

$$y = \frac{f - c}{b - e}.$$

Substituting this into Equation 1, we find that

$$x = -b\left(\frac{f - c}{b - e}\right) - c.$$

The assumption that $b \neq e$ leads to a contradiction: the intersection $\ell \cap m$ contains two distinct points, but an arbitrary point in the intersection is

$$\left(-b\left(\frac{f - c}{b - e}\right) - c, \frac{f - c}{b - e}\right).$$

Hence we reject the assumption that $b \neq e$. Since $b = e$, Equation 3 implies that $c = f$ as well; therefore, $\ell = m$.

Now let us consider the $e = 1$ case. By subtracting d times Equation 1 from Equation 2, we obtain the equation

$$(1 - db)y + (f - dc) = 0. \tag{4}$$

Assume $1 \neq db$. As in the previous case, we can solve for y in Equation 4, namely

$$y = \frac{dc - f}{1 - db}.$$

Once again this determines a unique value for x in Equation 1, a contradiction. Hence we reject the assumption that $1 \neq db$. Then $f = dc$ and so by Lemma 8,

$$m = L(d, 1, f) = L(d \cdot 1, d \cdot b, d \cdot c) = L(1, b, c) = \ell.$$

Since either $d = 1$ or $e = 1$, it follows that $\ell = m$. □

Proposition 13. *The cartesian plane satisfies Axiom 1.*

Proof. Let P and Q be distinct points. By Lemma 11, there exists a line ℓ on which these points lie. Suppose P and Q lie on the line m. Then $\ell \cap m$ contains two distinct points and so by Lemma 12, $\ell = m$. □

Proposition 14. *The cartesian plane satisfies Axiom 3.*

Proof. The cartesian plane contains the distinct points $P = (0, 0)$, $Q = (1, 0)$, and $R = (0, 1)$. One can verify the following:

- The unique line on which P and Q lie is $L(0, 1, 0)$, which does not include R.
- The unique line on which P and R lie is $L(1, 0, 0)$, which does not include Q.
- The unique line on which Q and R lie is $L(1, 1, 0)$, which does not include P.

Hence the points P, Q, and R are non-collinear. □

Theorem 15. *The cartesian plane is a model of incidence geometry.*

Proof. This follows immediately from Proposition 10, Proposition 13, and Proposition 14. □